TIGHT CONTACT STRUCTURES ON THE BRIESKORN SPHERES $-\Sigma(2,3,6n-1)$ AND CONTACT INVARIANTS

PAOLO GHIGGINI AND JEREMY VAN HORN-MORRIS

Abstract. We compute the Ozsváth–Szabó contact invariants for all tight contact structures on the manifolds $-\Sigma(2,3,6n-1)$ using twisted coefficient and a previous computation by the first author and Ko Honda. This computation completes the classification of the tight contact structures in this family of 3-manifolds.

1. Introduction

The family of 3-manifolds $-\Sigma(2,3,6n-1)$ defined by the surgery diagram in Figure 1 has been an exciting playground for contact topologists for many years, and any progress in the knowledge of the tight contact structures in this family has lead us to a progress in our understanding of three-dimensional contact topology.

These manifolds first were used by Lisca and Matić in [15] to give an example of the power of the recently discovered Seiberg–Witten invariants in distinguishing tight contact structures. Later Etnyre and Honda [2] proved that $-\Sigma(2,3,5)$ supports no tight contact structure, giving the first example of such a manifold. Tight contact structures on $-\Sigma(2,3,17)$ were instrumental in the first vanishing theorem for the Ozsváth–Szabó contact invariant in [5]. Finally the first author proved in [3] that $-\Sigma(2,3,6n-1)$ carries a strongly fillable contact structure which is not Stein fillable when $n \geq 3$, thus showing that strong and Stein fillability are different concepts in dimension three.

The goal of this paper is to give a complete classification of tight contact structures on manifolds in this family, and to do that we will compute their Ozsváth–Szabó contact invariants. The proof is a delicate computation using Heegaard Floer homology with twisted coefficients.

It has been known for a while that $-\Sigma(2,3,6n-1)$ supports at most $\frac{n(n-1)}{2}$ distinct contact structures up to isotopy. We will denote them by $\eta^n_{i,j}$ where $0 \leq i \leq n-2$ and $-n+i+2 \leq j \leq n-i-2$ with $j \equiv n-i \pmod 2$. The geometric meaning of the indices i and j will be explained in the next section. In order to simplify the exposition we define the following notation for the sets of indices of the contact structures $\eta^n_{i,j}$:

Definition 1.1. For any $n \geq 2$ we define

$$\mathcal{P}_n = \left\{ (i,j) \in \mathbb{Z} \times \mathbb{Z} : \begin{array}{l} 0 \leq i \leq n-2, \\ |j| \leq n-i-2 \text{ with } j \equiv n-i \ (\mathrm{mod}\, 2) \end{array} \right\}.$$

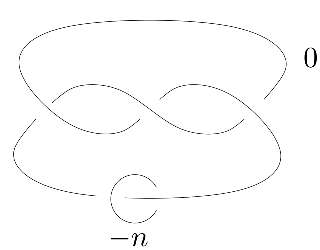

Figure 1. Surgery diagram for $-\Sigma(2,3,6n-1)$

We can visualize $\mathcal{P}_n$ (and the contact structures indexed by its elements) as a triangle with $n-1$ rows and $(n-2, 0)$ at its upper vertex. For example for $n = 5$ we have:

$$
\begin{array}{ccccccc}
 & & & \eta^5_{3,0} & & & \\
 & & \eta^5_{2,-1} & & \eta^5_{2,1} & & \\
 & \eta^5_{1,-2} & & \eta^5_{1,0} & & \eta^5_{1,2} & \\
\eta^5_{0,-3} & & \eta^5_{0,-1} & & \eta^5_{0,1} & & \eta^5_{0,3}
\end{array}
\tag{1}
$$

For any n, the contact structures on the bottom row (i.e. those with $i = 0$) are obtained by Legendrian surgery on all possible Legendrian realizations of the link in Figure 1 (see Figure 9), and therefore are Stein fillable. All other contact structures are strongly symplectically fillable, and the top one (i.e., $\eta^n_{n-2.0}$) is known not to be Stein fillable by [3]. No Stein filling is known for $\eta^n_{i,j}$ when $i > 0$, therefore we conjecture the following:

Conjecture 1.2. *The contact structures $\eta^n_{i,j}$ are not Stein fillable if $i > 0$.*

Now we can state the main result of this article:

Theorem 1.3. *Let $c(\eta^n_{i,j})$ denote the Ozsváth–Szabó contact invariant of $\eta^n_{i,j}$. We can choose representatives for $c(\eta^n_{0,j})$ such that, for any $(i, j) \in \mathcal{P}_n$, the contact invariant of $\eta^n_{i,j}$ is computed by the formula:*

$$
c(\eta^n_{i,j}) = \sum_{k=0}^{i} (-1)^k \binom{i}{k} c(\eta^n_{0,j-i+2k}).
\tag{2}
$$

We can reformulate Theorem 1.3 in plain English as follows. Any $(i, j) \in \mathcal{P}_n$ determines a sub-triangle $\mathcal{P}_n(i, j) \subset \mathcal{P}_n$ with top vertex at (i, j) defined as

$$
\mathcal{P}_n(i, j) = \{(k, l) \in \mathcal{P}_n : 0 \le k \le i \quad \text{and} \quad j - k \le l \le j + k\}\,.
$$

The contact invariant of $\eta^n_{i,j}$ is then a linear combination of the invariants of the contact structures parametrized by the pairs in the base of $\mathcal{P}_n(i, j)$. In order to compute the coefficients we associate natural numbers to the elements of $\mathcal{P}_n(i, j)$, starting by associating 1 to the vertex (i, j), and going downward following the rule of the Pascal triangle. Then the numbers associated to the elements in the bottom row, taken with alternating signs, are the coefficients of the contact invariants of the corresponding contact structures in the sum in Equation (2).

Olga Plamenevskaya proved in [23] that the contact invariants of the contact structures parametrized by the elements in the bottom row of $\mathcal{P}_n$ (i.e. those with $i = 0$) are linearly independent, so all $\eta^n_{i,j}$ have distinct contact invariants. Thus we have the following corollary:

Corollary 1.4. *$-\Sigma(2, 3, 6n - 1)$ admits exactly $\frac{n(n-1)}{2}$ distinct isotopy classes of tight contact structures with non zero and pairwise distinct Ozsváth–Szabó contact invariants.*

The same classification result could probably be derived also from Wu's work on Legendrian surgeries [25] and from the computation of the contact invariants with twisted coefficients of contact manifolds with positive Giroux's torsion in [6]. However it is not clear how to obtain a complete description of the contact invariants as in Theorem 1.3 from that approach.

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2. Contact structures on $-\Sigma(2,3,6n-1)$

2.1. **Construction of the tight contact structures.** We introduce the notation $Y_n = -\Sigma(2,3,6n-1)$, and, coherently with the standard surgery convention, we define Y_∞ to be the 3-manifold obtained by 0-surgery on the right-handed trefoil knot. We describe Y_∞ as a quotient of $T^2 \times \mathbb{R}$ (with coordinates (x,y) on T^2 and t on $\mathbb{R}$):

$$Y_\infty = T^2 \times \mathbb{R}/(\mathbf{v}, t) = (A\mathbf{v}, t-1)$$

where $A\colon T^2 \to T^2$ is induced by the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. In [8] Giroux constructed a family of weakly symplectically fillable contact structures ξ_i on Y_∞ for $i \geq 0$ as follows. For any $i \geq 0$, fix a function $\varphi_i\colon \mathbb{R} \to \mathbb{R}$ such that:

(1) $\varphi_i'(t) > 0$ for any $t \in \mathbb{R}$, and
(2) $2(n-1)\pi \leq \sup\limits_{t\in\mathbb{R}}(\varphi_i(t+1) - \varphi_i(t)) < (2n-1)\pi$.

By condition (1) the 1-form

$$\alpha_i = \sin(\varphi_i(t))dx + \cos(\varphi_i(t))dy$$

defines a contact structure $\tilde{\xi}_i = \ker \alpha_i$ on $T^2 \times \mathbb{R}$. Moreover it is possible to choose φ_i such that the contact structure $\tilde{\xi}_i$ (but not the 1-form α_i in general) is invariant under the action $(\mathbf{v}, t) \mapsto (A\mathbf{v}, t-1)$, therefore it defines a contact structure ξ_i on Y_∞.

Proposition 2.1 ([8])**.** *For any fixed i, the contact structure ξ_i is tight, and its isotopy class does not depend on the chosen function φ_i.*

The knot

$$F = \{\mathbf{0}\} \times \mathbb{R}/(\mathbf{0}, t) = (\mathbf{0}, t-1) \subset Y_\infty$$

is Legendrian with respect to ξ_i for any i. In [5] the first author proved the following properties of F:

Proposition 2.2 ([5, Lemma 3.5])**.** *There exists a framing on F such that:*

(1) $tn(F, \xi_i) = -i-1$
(2) *performing surgery on Y_∞ along F with surgery coefficient $-n$ yields Y_n.*

Moreover, even though F is essential, we can define a rotation number $\texttt{rot}(\mathcal{L}, \xi_i)$ for an oriented Legendrian knot $\mathcal{L} \subset (Y_\infty, \xi_i)$ smoothly isotopic to F by setting $\texttt{rot}(F, \xi_i) = 0$ for all i and defining $\texttt{rot}(\mathcal{L}, \xi_i) = \texttt{rot}(\mathcal{L} \cup \overline{F}, \xi_i)$, where $\overline{F}$ denotes F with the opposite orientation. We do not need to reference a Seifert surface for $\mathcal{L} \cup \overline{F}$ because $c_1(\xi_i) = 0$. We are finally in position to give a precise definition of the contact structures $\eta^n_{i,j}$ and, at the same time, to explain the topological meaning of the indices i and j.

Definition 2.3. For any $(i,j) \in \mathcal{P}_n$ the contact manifold $(Y_n, \eta^n_{i,j})$ is obtained by Legendrian surgery on (Y_∞, ξ_i) along a Legendrian knot $F_{i,j}$ which is obtained by applying $n-i-1$ stabilizations to F, choosing their signs so that $\texttt{rot}(F_{i,j}, \xi_i) = j$.

In order to complete the classification of tight contact structures on Y_n we need two steps:

(1) prove that there are at most $\frac{n(n-1)}{2}$ distinct tight contact structures on Y_n up to isotopy, and
(2) prove that the contact structures $\eta^n_{i,j}$ are all pairwise non isotopic.

The first step is a folklore result; it follows from the arguments of [7], nevertheless we are going to sketch its proof in the next sub-section. The second step is a corollary of Theorem 1.3, which will be proved in the last section.

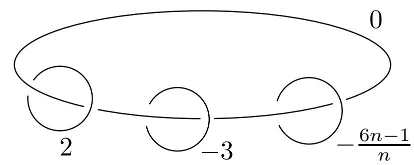


FIGURE 2. Surgery diagram for $-\Sigma(2,3,6n-1)$

2.2. **Upper bound.** The upper bound on the number of tight contact structures on Y_n can be easily obtained following the strategy in [7], where the tight contact structures on $-\Sigma(2,3,11)$ have been classified. In fact, the manifold denote by Y_n in this paper corresponds to the manifold denoted by $M(-\frac{1}{2}, \frac{1}{3}, \frac{n}{6n-1})$ in [7]. We recall the conventions of that paper.

The manifold Y_n can be described also by the surgery diagram shown in Figure 2. See [7, Figure 17] for a sequence of Kirby move from the diagram in Figure 2 to the diagram in Figure 1.

The surgery diagram 2 describes a splitting of Y_n into four pieces:

$$Y_n = (\Sigma \times S^1) \cup V_1 \cup V_2 \cup V_3$$

where Σ is a three-punctured sphere, i.e. a pair of pants, and V_1, V_2, and V_3 are solid tori. We orient the boundary of $\Sigma \times S^1$ by the "inward normal vector first" convention (i.e. we give it the opposite of the usual boundary orientation), and identify each component $\partial(S^1 \times \Sigma)_i$ of $\partial(S^1 \times \Sigma)$ with $\mathbb{R}^2/\mathbb{Z}^2$ by setting $\binom{0}{1}$ as the direction of the S^1-fibres and $\binom{1}{0}$ as the direction of $\partial(\{pt\} \times \Sigma)$.

We also fix identifications of ∂V_i with $\mathbb{R}^2/\mathbb{Z}^2$ by setting $\binom{1}{0}$ as the direction of the meridian. Then we obtain the manifold Y_n by attaching the solid tori V_i to $S^1 \times \Sigma$, where the attaching maps $A_i \colon \partial V_i \to \partial(S^1 \times \Sigma)_i$ are given by

$$A_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 6n-1 & 6 \\ -n & -1 \end{pmatrix}.$$

This construction induces a Seifert fibration on Y_n where the curves $S^1 \times \{pt\}$ are regular fibers, and the cores of the solid tori V_i are the singular fibers. The regular fibers have a natural framing coming from the Seifert fibration, and the singular fibers have a framing coming from the chosen identification of ∂V_i with $\mathbb{R}^2/\mathbb{Z}^2$. These framings can be extended in a unique way to all curves which are isotopic to fibers because the manifolds Y_n are integer homology spheres. Therefore — if we have a contact structure ξ on Y_n — we can speak about the twisting number $tn(L, \xi)$ of a Legendrian curve isotopic to a fiber of the Seifert fibration.

Definition 2.4. For any contact structure ξ on Y_n, we define the *maximal twisting number* of ξ as

$$t(\xi) = \max_{L \in \mathfrak{L}} \min\{tn(L, \xi), 0\}$$

where $\mathfrak{L}$ is the set of all Legendrian curves in Y_n which are smoothly isotopic to a regular fibre.

The maximal twisting number is clearly an isotopy invariant of the contact structure ξ.

Proposition 2.5. *Let ξ be a tight contact structure on Y_n. Then $t(\xi) < 0$.*

Proof. The proof is as in [7, Theorem 4.14]. □

Lemma 2.6. *If ξ can be isotoped so that the singular fiber F_2 is a Legendrian curve with twisting number $tb(F_2, \xi) = -1$, then there is a Legendrian regular fiber with twisting number zero. In particular ξ is overtwisted.*

Proof. We isotope F_1 so that it becomes a Legendrian curve with twisting number $tb(F_1) = k_1 \ll 0$. Let V_1 and V_2 be standard neighbourhoods of F_1 and F_2 respectively. We assume that $\partial(Y_n \setminus V_1)$

and $\partial(Y_n \setminus V_2)$ have Legendrian rulings of infinite slope, and take a convex annulus A with boundary on a Legendrian ruling curve of $\partial(Y_n \setminus V_1)$ and one of $\partial(Y_n \setminus V_2)$.

The slope of $\partial(Y_n \setminus V_1)$ is $\frac{k_1}{2k_1-1}$, and the slope of $\partial(Y_n \setminus V_2)$ is $-\frac{1}{2}$. As long as $k_1 \le -1$ the Imbalance Principle [11, Proposition 3.17] provides a bypass along a Legendrian ruling of $\partial(Y_n \setminus V_1)$, therefore we can apply the Twisting Number Lemma [11, Lemma 4.4] to increase the twisting number k_1 of a singular fibre by one up to $k_1 = 0$, which corresponds to slope 0 on $\partial(Y_n \setminus V_1)$. At this point there are two possibilities for the annulus A between $\partial(Y_n \setminus V_1)$ and $\partial(Y_n \setminus V_2)$: either A carries a bypass for $\partial(Y_n \setminus V_1)$, or it does not. If such a bypass exists, then the slope of $\partial(Y_n \setminus V_1)$ can be made infinite, and we are done. If there is no such a bypass, cutting along A and rounding the edges yields a torus with slope 0 (see [11, Lemma 3.11]), which is $-n$ when measured in ∂V_3. In this case by [11, Proposition 4.16] we find a convex torus in V_3 with slope $-n + \frac{1}{6}$, which corresponds to infinite slope in $\partial(M \setminus V_3)$. □

Proposition 2.7. *Let ξ be a tight contact structure on Y_n. Then $t(\xi) = -6k + 1$ for some k with $0 < k < n - 1$.*

Proof. Let $t(\xi) = -q$. We start by assuming that the contact structure has been isotoped so that there is a Legendrian regular fibre L with twisting number $tb(L, \xi) = -q$, and the singular fibres F_i are Legendrian curves with twisting numbers $k_i \ll 0$. We take V_i to be a standard neighbourhood of the singular fiber F_i disjoint from L for $i = 1, 2, 3$.

The slopes of ∂V_i are $\frac{1}{k_i}$, while the slopes of $\partial(Y_n \setminus V_i)$ are $\frac{k_1}{2k_1-1}$, $-\frac{k_2}{3k_2+1}$, and $-\frac{nk_3+1}{(6n-1)k_3+6}$ respectively. We also assume that the Legendrian rulings on each $\partial(Y_n \setminus V_i)$ have infinite slope, and take convex annuli A_i whose boundary consists of L and of a Legendrian ruling curve of $\partial(Y_n \setminus V_i)$ for $i = 1, 2$. If $2k_1 - 1 < -q$ the Imbalance Principle [11, Proposition 3.17] provides a bypass along a Legendrian ruling curve either in $\partial(Y_n \setminus V_1)$ or in $\partial(Y_n \setminus V_2)$, then we can apply the Twisting Number Lemma [11, Lemma 4.4] to increase the twisting number k_i of a singular fibre by one until either $2k_1 - 1 < -q$, or $k_1 = 0$. Similarly we use the annulus A_2 in the same way to increase k_2 until either $3k_2 + 1 = -q$, or $k_2 = -1$.

If $2k_1 - 1 = 3k_2 + 1 = -q$ we can write $k_1 = -3k + 1$, $k_2 = -2k$, and $q = 6k - 1$ for some $k > 0$. Take a convex annulus A with Legendrian boundary consisting of a Legendrian ruling curve of $\partial(Y_n \setminus V_1)$ and of one of $\partial(Y_n \setminus V_2)$. The dividing set of A contains no boundary parallel arc, otherwise we could attach a bypass to either $\partial(Y_n \setminus V_1)$ or to $\partial(Y_n \setminus V_2)$, and the vertical Legendrian ruling curves of the resulting torus would contradict the maximality of $-q$. If we cut $Y_n \setminus (V_1 \cup V_2)$ along A and round the edges, we obtain a torus with slope $-\frac{k}{6k+1}$ isotopic to $\partial(Y_n \setminus V_3)$. Its slope corresponds to $-n + k$ on ∂V_3. If $k \ge n$ we can find a standard neighbourhood V_3' of F_3 with infinite boundary slope by [11, Proposition 4.16]. This boundary slope becomes $-\frac{1}{6}$ if measured with respect to $\partial(Y_n \setminus V_3')$, contradicting $q > 6n - 1$ (remember that we are assuming $n \ge 2$). □

Proposition 2.8. *There are at most $\frac{n(n-1)}{2}$ isotopy classes of tight contact structures on Y_n.*

Proof. If $t(\xi) = -6k + 1$ we can find a neighbourhood V_3 of F_3 such that $\partial(Y_n \setminus V_3)$ has slope $-\frac{k}{6k+1}$. This slope corresponds to $-n + k$ on ∂V_3. By the classification of tight contact structures on solid tori [11, Theorem 2.3], there are $n - k$ tight contact structures on V_3. Since k ranges from 1 to $n - 1$, we have a total count of at most $\frac{n(n-1)}{2}$ tight contact structures on Y_n. □

3. Heegaard Floer homology with twisted coefficients

In the computation of the contact invariants of $\eta_{i,j}$ we will use the Heegaard Floer homology groups with twisted coefficients. Since this theory is not as well known as the usual Heegaard Floer theory, we give a brief review of its properties. A more detailed exposition for the interested reader can be found in the original papers [21, 18] and in [13].

Let Y be a closed, connected and oriented 3-manifold. In the following, singular cohomology groups will always be taken with integer coefficients, unless a different Abelian group is explicitly indicated. Given a module M over the group algebra $\mathbb{Z}[H^1(Y)]$ and a Spin^c-structure $\mathfrak{t} \in \mathrm{Spin}^c(Y)$, in [21, Section 8] Ozsváth and Szabó defined the Heegaard Floer homology group with twisted coefficients $\underline{\widehat{HF}}(Y, \mathfrak{t}; M)$, which has a natural structure of a $\mathbb{Z}[H^1(Y)]$-module. When we omit the Spin^c-structure from the notation, we understand that we take the direct sum over all Spin^c-structures of Y.

Two modules M are of particular interest: the free module of rank one $M = \mathbb{Z}[H^1(Y)]$, and the module $M = \mathbb{Z}$ with the trivial action of $H^1(Y)$. In the first case we will denote $\underline{\widehat{HF}}(Y; M) = \underline{\widehat{HF}}(Y)$, and in the second case $\underline{\widehat{HF}}(Y; M) = \widehat{HF}(Y)$.

To a cobordism W from Y_0 to Y_1, in [18] Ozsváth and Szabó associated morphisms between the Heegaard Floer homology groups with twisted coefficients. However there is an extra complication which is absent in the untwisted case: the groups are usually modules over different rings, and we need to define a "canonical" way to transport coefficients across a cobordism.

The automorphism $x \mapsto -x$ of $H^1(Y)$ induces an involution of $\mathbb{Z}[H^1(Y)]$ which we call *conjugation* and denote by $r \mapsto \bar{r}$. If M is a module over $\mathbb{Z}[H^1(Y)]$, we define a new module $\overline{M}$ by taking M as an additive group, and equipping it with the multiplication $r \otimes m \mapsto \bar{r} \cdot m$.

Let us define

$$K(W) = \ker\left(H^2(W, \partial W) \to H^2(W)\right).$$

Its group algebra $\mathbb{Z}[K(W)]$ has the structure of both a $\mathbb{Z}[H^1(Y_0)]$-module and a $\mathbb{Z}[H^1(Y_1)]$-module induced by the connecting homomorphism $\delta \colon H^1(\partial W) \to H^2(W, \partial W)$ for the relative long exact sequence of the pair $(W, \partial W)$, therefore we can define the $\mathbb{Z}[H^1(Y_1)]$-module $M(W)$ as

$$M(W) = \overline{M} \otimes_{\mathbb{Z}[H^1(Y_0)]} \mathbb{Z}[K(W)].$$

Theorem 3.1 ([18, Theorem 3.8]). *To any cobordism W from Y_0 to Y_1 with a* Spin^c*-structure $\mathfrak{s} \in \mathrm{Spin}^c(W)$, we associate an anti-$\mathbb{Z}[H^1(Y_0)]$-linear map*

$$F_{W,\mathfrak{s}} \colon \underline{\widehat{HF}}(Y_0, \mathfrak{s}|_{Y_0}; M) \to \underline{\widehat{HF}}(Y_1, \mathfrak{s}|_{Y_1}; M(W))$$

which is well defined up to sign, right multiplication by invertible elements of $\mathbb{Z}[H^1(Y_1)]$, and left multiplication by invertible elements of $\mathbb{Z}[H^1(Y_0)]$.

We will denote the equivalence class of such a map by $[F_{W,\mathfrak{s}}]$.

Remark 3.2. Let $\iota_0 \colon Y_0 \to W$ and $\iota_1 \colon Y_1 \to W$ be the inclusions. If the maps $(\iota_0)_* \colon H_2(Y_0) \to H_2(W)$ and $(\iota_1)_* \colon H_2(Y_1) \to H_2(W)$ are injective and $\mathrm{Im}(\iota_0)_* = \mathrm{Im}(\iota_1)_*$ we can define an isomorphism $(\iota_W)_* = (\iota_1)_*^{-1}(\iota_0)_* \colon H_2(Y_0) \to H_2(Y_1)$. After composing with Poincaré dualities, we obtain an isomorphism

$$\iota_W^! \colon H^1(Y_0) \to H^1(Y_1),$$

which induces a structure of $\mathbb{Z}[H^1(Y_0)]$-module on $\mathbb{Z}[H^1(Y_1)]$. Moreover with this structure there is an anti-$\mathbb{Z}[H^1(Y_0)]$-linear isomorphism

$$\overline{\mathbb{Z}[H^1(Y_0)]} \otimes_{\mathbb{Z}[H^1(Y_0)]} \mathbb{Z}[K(W)] \cong \mathbb{Z}[H^1(Y_1)],$$

and then a $\mathbb{Z}[H^1(Y_0)]$-*linear* cobordism map

$$F_W \colon \underline{\widehat{HF}}(Y_0) \to \underline{\widehat{HF}}(Y_1).$$

Let us decompose $\delta \colon H^1(\partial W) = H^1(Y_0) \oplus H^1(Y_1) \to K(W)$ as $\delta = \delta_0 \oplus \delta_1$, where each $\delta_i \colon H^1(Y_i) \to K(W)$ is an isomorphism. We have a commutative diagram:

$$\begin{tikzcd} H^1(Y_0) \arrow[rr, "-\iota^!_W"] \arrow[dr, "\delta_1"'] & & H^1(Y_1) \arrow[dl, "\delta_2"] \\ & K(W) & \end{tikzcd}$$

In fact, let c_0 be the Poincaré dual of $a \in H^1(Y_0)$, and c_1 be the Poincaré dual of $\iota^!_W(a) \in H^1(Y_1)$. Then $(\iota_1)_*(c_1) - (\iota_0)_*(c_0) = 0$ in $H_2(W)$, which implies that $c_1 - c_0 = \partial C$ for some class $C \in H_3(W, \partial W)$. Taking Poincaré duals on W and ∂W we obtain that $a + \iota^!_W(a)$ — the change of sign because W induces the opposite orientation on Y_0 — is in the image of the restriction map $H^1(W) \to H^1(\partial W)$, therefore $\delta(a + \iota^!_W(a)) = 0$, from which the commutativity of the diagram follows.

The isomorphism between $\overline{\mathbb{Z}[H^1(Y_0)]} \otimes_{\mathbb{Z}[H^1(Y_0)]} \mathbb{Z}[K(W)]$ and $\mathbb{Z}[H^1(Y_1)]$ is given by the map

$$e^a \otimes e^b \mapsto e^{\iota^!_W(a) + \delta_2^{-1}(b)}.$$

This map is well defined by the universal property of the tensor product, because it is induced by a $\mathbb{Z}[H^1(Y_0)]$-bilinear map

$$\overline{\mathbb{Z}[H^1(Y_0)]} \times \mathbb{Z}[K(W)] \to \mathbb{Z}[H^1(Y_0)].$$

In fact for all $a' \in H^1(Y_0)$ we have

$$(e^{a-a'}, e^b) \mapsto e^{\iota^!_W(a-a') + \delta_2^{-1}(b)}$$
$$(e^a, e^{\delta_1(a')+b}) \mapsto e^{\iota^!_W(a) + \delta_2^{-1}(\delta_1(a')+b)},$$

but $e^{\iota^!_W(a-a') + \delta_2^{-1}(b)} = e^{-a'} \cdot e^{\iota^!_W(a) + \delta_2^{-1}(b)}$.

The cobordism maps fit into surgery exact sequences, of which we state only the simple case we use in the paper.

Theorem 3.3 ([21, Theorem 9.21]; cf. [13, Section 9])**.** *Let Y be an integer homology sphere, and K be a knot in it. We identify framings on K to integer numbers by assigning 0 to the framing induced by an embedded surface with boundary in K, and denote by $Y_n(K)$ the manifold obtained by performing n surgery along K. Then there is an exact triangle*

$$\begin{tikzcd} \widehat{HF}(Y)[t^{-1}, t] \arrow[rr, "F"] & & \widehat{HF}(Y_{-1}(K))[t^{-1}, t] \arrow[dl] \\ & \underline{\widehat{HF}}(Y_0(K)) \arrow[ul] & \end{tikzcd}$$

If W is the 4-dimensional cobordism from Y to $Y_{-1}(K)$ induced by the surgery, $[\widehat{\Sigma}]$ is a generator of $H_2(W)$, and $\mathfrak{s}_k$ is the unique Spin^c*-structure on W such that $\langle c_1(\mathfrak{s}_k), [\widehat{\Sigma}]\rangle = 2k+1$, then*

$$F = \sum_{k \in \mathbb{Z}} F_{W, \mathfrak{s}_k} \otimes t^k.$$

Maps between Heegaard Floer homology groups with twisted coefficients satisfy composition formulas which are both more involved and more powerful than the analogous formulas for ordinary coefficients. The source of the difference is that, given cobordisms W_0 from Y_0 to Y_1 and W_1 from Y_1 to Y_2, the coefficient ring $M(W)$ associated to the map F_W induced by the composite cobordism $W = W_1 \cup W_0$ is usually smaller than the coefficient ring $M(W_0)(W_1)$ associated to the composition $F_{W_1} \circ F_{W_0}$. More precisely:

Lemma 3.4. *There is an exact sequence*

$$0 \to K(W) \xrightarrow{\iota} \frac{K(W_0) \oplus K(W_1)}{H^1(Y_1)} \to \mathrm{Im}(\delta) \to 0$$

where $\delta\colon H^1(Y_1) \to H^2(W)$ *is the connecting homomorphism for the Mayer–Vietoris sequence of the triple* (W_0, W_1, W)*.*

Proof. It follows from the commutative diagram:

$$\begin{tikzcd} & H^2(W_0) \oplus H^2(W_1) & H^2(W) \arrow[l] & H^1(Y_1) \arrow[l, "\delta"'] \\ H^1(Y_1) \arrow[r] & H^2(W_0, \partial W_0) \oplus H^2(W_1, \partial W_1) \arrow[u] \arrow[r] & H^2(W, \partial W) \arrow[u] & \end{tikzcd}$$

where the top row is the Mayer–Vietoris sequence and the bottom row is the relative cohomology sequence for the pair (W, Y_1). In fact $H^2(W_0, \partial W_0) \oplus H^2(W_1, \partial W_1) \cong H^2(W, Y_0)$ by homotopy equivalence and excision. □

The inclusion $\iota\colon K(W) \to \dfrac{K(W_0) \oplus K(W_1)}{H^1(Y_1)}$ gives rise to a projection

$$\Pi\colon \mathbb{Z}[K(W_0)] \otimes_{\mathbb{Z}[H^1(Y_1)]} \mathbb{Z}[K(W_1)] \cong \mathbb{Z}\left[\frac{K(W_0) \oplus K(W_1)}{H^1(Y_1)}\right] \longrightarrow \mathbb{Z}[K(W)]$$

defined by

$$\Pi(e^w) = \begin{cases} e^w & \text{if } w = \iota(v) \text{ for some } v \\ 0 & \text{otherwise.} \end{cases}$$

which extends to a projection $\Pi_M\colon M(W_0)(W_1) \to M(W)$ for any $\mathbb{Z}[H^1(Y_0)]$-module M. The composition law for twisted coefficients can be stated as follows.

Theorem 3.5 ([18, Theorem 3.9]; cf. [13, Theorem 2.9])**.** *Let* $W = W_0 \cup_{Y_1} W_1$ *be a composite cobordism with a* Spin^c*-structure* $\mathfrak{s}$*. Write* $\mathfrak{s}_i = \mathfrak{s}|_{W_i}$*. Then there are choices of representatives for the maps* F_{W_0}*,* F_{W_1}*, and* F_W *such that:*

$$[F_{W,\mathfrak{s}+\delta h}] = [\Pi_M \circ F_{W_2,\mathfrak{s}_2} \circ e^{-h} \cdot F_{W_1,\mathfrak{s}_1}]$$

where $h \in H^1(Y_1)$ *and* $\delta\colon H^1(Y_1) \to H^2(W)$ *is the connecting homomorphism for the Mayer–Vietoris sequence.*

To a contact structure ξ on Y we can associate an element $c(\xi, M) \in \underline{\widehat{HF}}(-Y, \mathfrak{t}_\xi)$ where $-Y$ denotes Y with the opposite orientation, and $\mathfrak{t}_\xi$ denotes the canonical Spin^c-structure on Y determined by ξ. This contact element is well defined up to sign and multiplication by invertible elements in $\mathbb{Z}[H^1(Y)]$, and we will denote $[c(\xi, M)]$ its equivalence class. When M is clear from the context we will drop it from the notation.

Theorem 3.6 (Ozsváth–Szabó [22])**.** *Let* ξ *be a contact structure on* Y*. Then:*

(1) $[c(\xi, M)]$ *is an isotopy invariant of* ξ*,*
(2) *if* $c_1(\xi)$ *is a torsion cohomology class, then* $[c(\xi, M)]$ *is a set of homogeneous elements of degree* $-\frac{\theta(\xi)}{4} - \frac{1}{2}$*, where* θ *is Gompf's* 3*-dimensional homotopy invariant defined in* [9, Definition 4.2]*,*
(3) *if* ξ *is overtwisted, then* $[c(\xi, M)] = \{0\}$*.*

The behaviour of the contact invariant is contravariant with respect to Legendrian surgeries, as described by the following theorem:

Theorem 3.7. *Let (Y_0,ξ_0) and (Y_1,ξ_1) be contact manifolds, and let (W,J) be a Stein cobordism from (Y_0,ξ_0) to (Y_1,ξ_1) obtained by Legendrian surgery on some Legendrian link in Y_0. If $\mathfrak{k}$ is the canonical* Spin^c*-structure on W for the complex structure J, then:*

$$[F_{W,\mathfrak{s}}(c(\xi',M))] = \begin{cases} [c(\xi,M(W))] \text{ if } \mathfrak{s}=\mathfrak{k} \\ \{0\} \text{ if } \mathfrak{s}\neq\mathfrak{k} \end{cases}$$

This theorem is essentially due to Ozsváth–Szabó [22], and an explicit statement has been given by Lisca–Stipsicz [16]. Here we state a generalization of [5, Lemma 2.10] to twisted coefficient and to links with more than one component. However the proof remains unchanged. In Theorem 3.7 the map $F_{W,\mathfrak{s}}$ is actually induced by the *opposite* cobordism which goes from $-Y'$ to $-Y$, and which is often denoted by $\overline{W}$. We chose to drop this extra decoration from the notation because, in the computation of the Ozsváth–Szabó invariants, our maps will *always* be induced by the opposite of the cobordisms constructed by Legendrian surgeries.

For any contact structure ξ on a 3–manifold Y we denote by $\overline{\xi}$ the contact structure on Y obtained from ξ by inverting the orientation of the planes. This operation is called *conjugation.* In Heegaard Floer homology there is an involution $\mathfrak{J}:\widehat{HF}(-Y)\to\widehat{HF}(-Y)$ defined in [21, Section 2.2] and [18, Section 5.2] which is closely related to conjugation of contact structures. We are going to state and use its main property only for the untwisted version of the contact invariant.

Theorem 3.8 ([5, Theorem 2.10])**.** *Let (Y,ξ) be a contact manifold, then*

$$c(\overline{\xi}) = \mathfrak{J}(c(\xi)).$$

4. Computation of the Ozsváth–Szabó contact invariants

We are going to sketch the strategy of the computation as a guide for the reader. The topological input is a Legendrian surgery along a Legendrian link $\mathcal{L}\cup\mathcal{C}$ which takes the contact manifold (Y_∞,ξ_{i+1}) to $(Y_n,\eta^n_{i,j})$. This Legendrian surgery factors in two ways, one through (Y_∞,ξ_i) and one through $(Y_{n+1},\eta^{n+1}_{i+1,j})$, depending on whether we perform the surgery first along $\mathcal{L}$, and then along $\mathcal{C}$, or vice versa. The knot $\mathcal{L}$ is a stabilization of F and $\mathcal{C}$ is a link which is naturally Legendrian in each (Y_∞,ξ_i) for $i>0$.

Then we have homomorphisms in Heegaard Floer homology mapping the invariants of the tight contact structures on Y_n to the invariants of the tight contact structures on Y_{n+1} above the bottom row of the triangle $\mathcal{P}_{n+1}$. We compute these invariants by an inductive argument using the fact that the invariants of the tight contact structures on the bottom row span $\widehat{HF}(-Y_n)$ in the appropriate degree [23].

A feature of the computation is that it requires the use of Heegaard Floer homology with twisted coefficients, namely the contact invariants of ξ_i with twisted coefficients, which were computed in [6]. This is somewhat surprising as the manifolds Y_n are integer homology spheres.

4.1. **The surgery construction.** We find the Legendrian link $\mathcal{L}\cup\mathcal{C}$ by studying open book decompositions of (Y_∞,ξ_i) and $(Y_n,\eta^n_{i,j})$. All knots will be oriented and $\overline{K}$ will be used to denote K with its orientation reversed.

Proposition 4.1. *There is a Legendrian link $\mathcal{L}_{l,r}\cup\mathcal{C}$ in (Y_∞,ξ_{i+1}), for every $i>0$, so that Legendrian surgery along $\mathcal{L}_{l,r}\cup\mathcal{C}$ gives the contact manifold $(Y_n,\eta^n_{i,j})$, while surgery along $\mathcal{L}_{l,r}$ gives the contact manifold $(Y_{n+1},\eta^{n+1}_{i+1,j})$, where $j=l-r$ and $n=l+r+i+2$. For every i, the links $\mathcal{L}_{l,r}\cup\mathcal{C}$ are smoothly isotopic in Y_∞. Further, the image of $\mathcal{L}_{l,r}$ in ξ_{i+1} under the surgery along $\mathcal{C}$ can be identified with $\mathcal{L}_{l,r}$ in ξ_i.*

We will prove Proposition 4.1 by first constructing open book decompositions compatible with (Y_∞,ξ_{i+1}) which have the Legendrian knot F sitting naturally on a page, and see how to stabilize

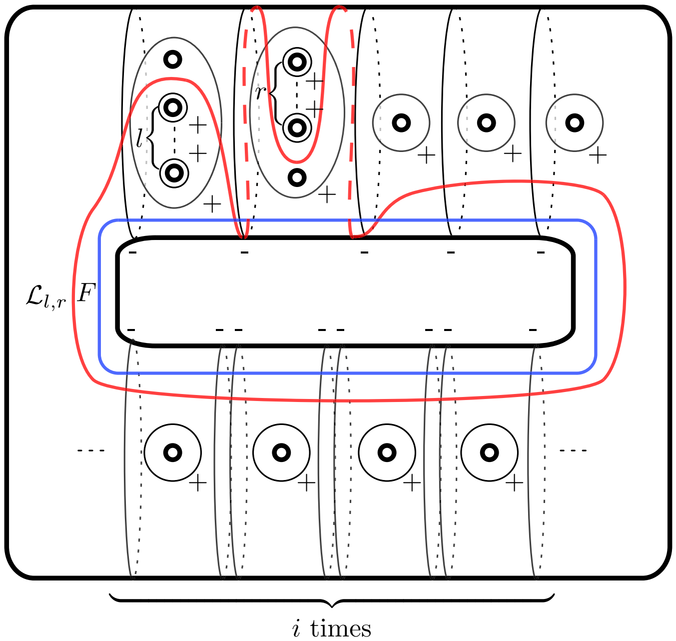


FIGURE 3. Genus one open book decompositions of the contact manifolds (Y_∞, ξ_i) and $(Y_n, \eta^n_{i,j})$ where $j = l - r$ and $n = l + r + i + 2$. The thick circles are the boundary of the page and the label i determines how many times the region along the bottom should be repeated. To get an open book compatible with (Y_∞, ξ_i), take the monodromy as the product of Dehn twists along all curves, except F and $\mathcal{L}_{l,r}$, with signs as indicated. Add a positive twist along the curve $\mathcal{L}_{l,r}$ to get $(Y_n, \eta_{i,j})$. The curve F is shown for comparison.

F to get the knot $\mathcal{L}_{l,r}$, still sitting on the page of a compatible open book. We then show how to modify this open book by adding positive Dehn twists to its monodromy to get an open book compatible with ξ_i, noting that this takes the knot F in ξ_i to the knot F in ξ_{i-1}.

We deal primarily with open books in their abstract form: as a surface S with boundary together with a self-diffeomorphism ϕ, usually presented as a product of Dehn twists along curves signed and labeled on a diagram of S. In Figure 3, such a diagram is given for the contact manifold (Y_∞, ξ_i). The surface S is a torus with many open disks removed, and the monodromy consists of positive (right-handed) Dehn twists about circles parallel to (most) boundary circles of S together with negative (left-handed) Dehn twists about certain meridians of the torus. In [24], it was shown how such open books corresponded to torus bundles and particular embeddings determining them. For convenience, we discuss some of that procedure here.

Begin by looking at an essential curve c on the torus S which is disjoint from all of the curves used in the presentation of the monodromy. (In Figure 3,c is a meridians of the torus, and intersects only the curves labeled F or $\mathcal{L}_{l,r}$.) Since c is fixed by the monodromy, it traces out a torus which will be a fiber in the bundle. As we move c around the torus page it traces out a family of torus fibers. Crossing a negative Dehn twists along a curve parallel to c induces a negative Dehn twist along the torus fiber, along a curve parallel to the page of the open book. Crossing a boundary circle with a positive Dehn twist induces a negative Dehn twist along the fiber, this time along a direction orthogonal to that of the page (see [24, Section 4.2]). These two Dehn twists correspond to the standard Dehn twist generators of the mapping class group of the torus and allow one to construct all the universally tight, linearly twisting contact structures on torus bundles (i.e., those described in Proposition 2.1).

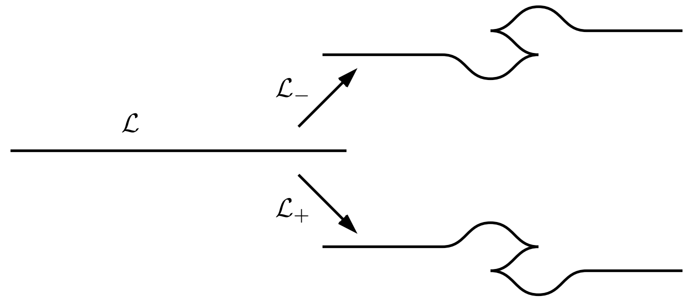


FIGURE 4. Local picture of the front projection for two Legendrian stabilizations.

The region above the bracket labeled 'i times' shows an open book compatible with a region of Giroux torsion one (multiplied i times). When $i = 0$, the open book describes the unique Stein fillable contact structure (see [24, Section 4.5]). Each of the pieces of the open book swept out as the meridional circle c passes a boundary component is a bypass and is compatible with a linear contact form of type used in Proposition 2.1. The horizontal arcs that make up the curve F in Figure 3 are linear in these local models (see [24, Figure 4.5]) and correspon to a Legendrian $\{pt\} \times I$ in $T^2 \times I$. Section 4.5 of [24] constructs our particular open books and shows they are compatible with the given contact structures; the manifold Y_∞ is the torus bundle with monodromy $T^{-1}S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. (This is different than what is stated in [24, Section 4.7.2]. The second author wrongly gave there a description of the torus bundle obtained from 0-surgery on the left-handed trefoil.)

We well prove this compatibility later in Proposition 4.6. The first thing we need for Proposition 4.1 is a way to stabilize F on the page of a compatible open book.

Definition 4.2. Let $\mathcal{L}$ be a Legendrian knot. There are two stabilizations of $\mathcal{L}$, positive and negative, denoted by $\mathcal{L}_+$ and $\mathcal{L}_-$, resp., given by the front projections shown in Figure 4.

Oriented to the right as shown in Figure 4, $\mathcal{L}_-$ and $\mathcal{L}_+$ have

$$\texttt{rot}(\mathcal{L}_- \cup \overline{\mathcal{L}}) = -1$$

and

$$\texttt{rot}(\mathcal{L}_+ \cup \overline{\mathcal{L}}) = +1.$$

Definition 4.3. Let L be a knot on a page of an open book. There are two stabilizations, left and right, denoted by L_l and L_r, resp., given by the local pictures shown in Figure 5.

We need the following lemma and will sketch its proof. A more complete proof can be found in [17].

Lemma 4.4. *Let K be a non-isolating knot on a page of an open book. If $\mathcal{L}$ is the Legendrian realization of K, then the Legendrian realization of K_r is the negative stabilization $\mathcal{L}_-$ and the Legendrian realization of K_l is the positive stabilization $\mathcal{L}_+$.*

Sketch of proof. First, observe that we can make K, K_l and K_r simultaneously Legendrian while sitting on the same page Σ of the open book, and we let $\mathcal{L}$, $\mathcal{L}_l$ and $\mathcal{L}_r$ refer to these Legendrian knots. All three knots are smoothly isotopic and so $K \cup \overline{K}_l$ bounds an annulus, A. We can make this annulus convex with Legendrian boundary $\mathcal{L} \cup \overline{\mathcal{L}}_l$, starting with the patch P, a subset of the page Σ, as shown in Figure 5. Since the dividing set is empty on Σ, it is empty on P and so the dividing set of A consists of boundary parallel arcs adjacent to K_l. Comparing framings shows that there is only one such arc and so $\texttt{rot}(K \cup \overline{K}_l) = \xi(A_+) - \xi(A_-) = -1$. This shows that K_l is the positive stabilization. $\square$

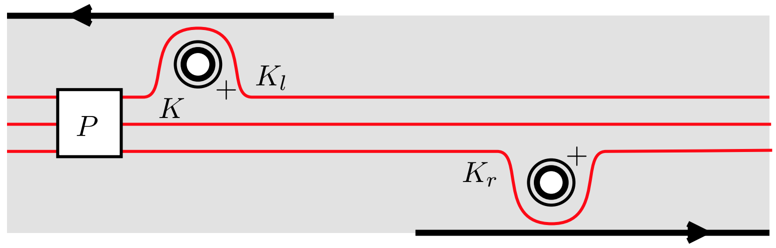


FIGURE 5. Local picture of the two stabilizations of a knot on a page of an open book. All knots K, K_l and K_r are oriented to the right, so that K_l is obtained by sliding K across a trivial stabilization on its left, and similarly K_r on its right. The patch P is used in Lemma 4.4 to prove that open book stabilizations give rise to Legendrian stabilizations. The embedded knots K, K_l and K_r are smoothly isotopic.

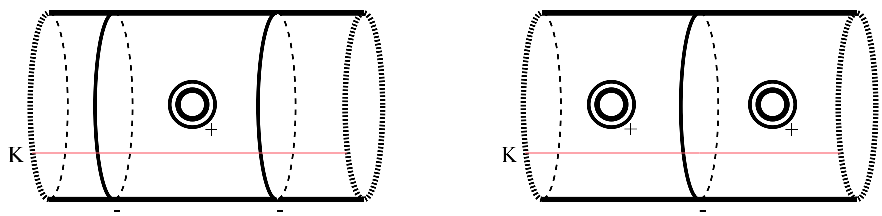


FIGURE 6. Local pictures B_1 and B_2 of open books which differ by a Hopf stabilization.

We will also need the following tool regarding stabilizations of open books.

Lemma 4.5 (The braid relation)**.** *Two open books which locally differ as in Figure 6 are related by a positive Hopf stabilization. There is a contact structure ξ defined in a neighborhood of the local picture compatible with both open books and such that the horizontal arc K is Legendrian and sitting on a page in each.*

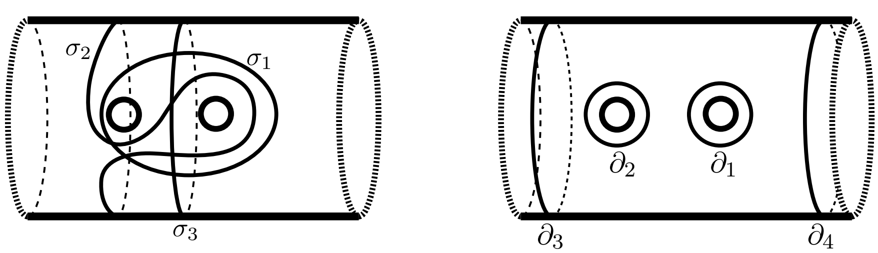


FIGURE 7. The lantern relation

Proof. The lantern relation (shown in Figure 7) relates the product of right-handed Dehn twists along each boundary component to the product of those along the three interior curves: $\partial_1\partial_2\partial_3\partial_4 = \sigma_1\sigma_2\sigma_3$, (where the Dehn twists act left to right). This diagram is different than the usual presentation of the lantern relation which draws the surface as a three-holed disk with the curves σ_i placed symmetrically, cf. [1, 14], but is more convenient for our purposes here.

The segment B_2 of the open book shown on the right hand side of Figure 6 is a 4-holed sphere with monodromy $\partial_1\partial_2\sigma_3^{-1}$ using the same curves as in Figure 7. After applying the lantern relation to B_2 we get the presentation shown in Figure 8 with an obvious destabilizing arc. After destabilizing, you are left with the open book segment B_1. Notice that the destabilizing arc is disjoint from the horizontal arc K shown in Figure 6 and so we can apply the braid relation even when there are

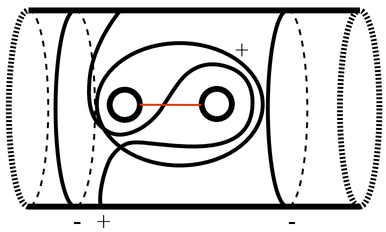


FIGURE 8. The monodromy of B_2 after applying the lantern relation. The obvious destabilizing arc is shown.

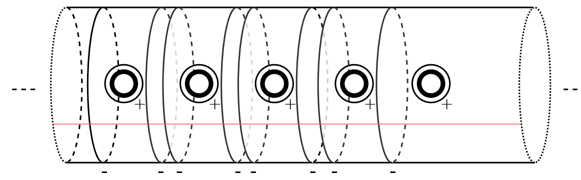
Dehn twists along curves running parallel to the segment, so long as the Dehn twists along ∂_1 and ∂_2 occur simultaneously in the described factorization. It was shown in [24, Section 4.2] how to construct a contact form compatible with B_1. Gluing two of these together gives a contact form compatible with both B_2 and B_1 and with the horizontal arc K being Legendrian and sitting on pages of each. □

Proposition 4.6. *The open books described in Figure 3 are compatible with the contact structures ξ_i on Y_∞ with the Legendrian knot F from Proposition 2.1 sitting on the page as the knot F in the figure.*

Proof. This can and is proved without resorting to many of the intricacies discussed in the beginning of the section. We first note that the region labeled i times is a region of Giroux torsion 1, multiplied i times, and for convenience, let us denote the associated open book $\mathfrak{B}_i$. From [24, Lemma 4.4.4] and its corollary, we see that the compatible contact structures are weakly fillable for all i and hence by the classification in [12] must be in the Giroux's family of tight contact structures constructed in Section 2.1. From [24, Section 4.5] (recalling that the monodromy for the right-handed trefoil is $T^{-1}S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$), we see that when $i = 0$, the compatible contact structure has zero Giroux torsion (indeed, it can be realized by Legendrian surgery on the Stein fillable contact structure on T^3). Thus $\mathfrak{B}_i$ is compatible with ξ_i. To see that the curve F in the diagram really is the Legendrian F discussed after Proposition 2.1, we do need a bit of detail. Looking at [24, Figure 4.5], the embedded diagram of a basic slice is compatible with a linearly twisting contact form of the type discussed in Proposition 2.1, and further the arc tangent to the t-axis at the left and right sides of the picture is Legendrian. One can glue any number of basic slices together (as well as gluing the front and back boundaries together) and the resulting open book will still be compatible with a linear contact form and the matched horizontal arc will remain Legendrian. □

Proof of Proposition 4.1. From Proposition 4.6 and Lemma 4.4 we see that $\mathcal{L}_{l,r}$ is $F_{i,j}$ from Section 2.1, (where $j = l - r$ and $n = l + r + i + 2$). Thus adding a right handed Dehn twist to the open book along $\mathcal{L}_{l,r}$ gives an open book compatible with $(Y_n, \eta^n_{i,j})$ and this describes the surgery from Y_∞ to Y_n.

To find the second link $\mathcal{C}$ and prove the lemma, we add positive twists to the monodromy of (Y_∞, ξ_{i+1}) in a small standard region of a compatible open book and see that, after applying the braid relation of Lemma 4.5, we have an open book compatible with (Y_∞, ξ_i). Notice now that each open book for (Y_∞, ξ_{i+1}), $i \geq 0$, has a region:



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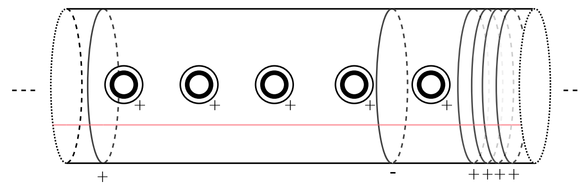
which describes a region of Giroux torsion one plus a basic slice. To this, we add positive twists to the monodromy to get to the open book



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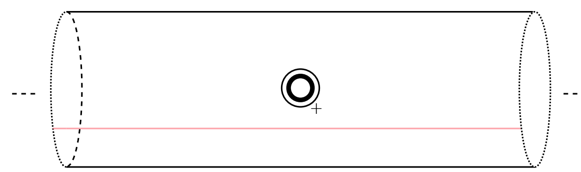
These positive twists make up the Legendrian link $\mathcal{C}$. Now repeatedly apply the braid relation and reduce to the open book for (Y_∞, ξ_i) where the Giroux torsion has been excised. Locally we now have



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Since the braid relation can be applied in the presence of Dehn twists along a horizontal curve, attaching Stein handles along $\mathcal{L}_{l,r} \cup \mathcal{C}$ in (Y_∞, ξ_{i+1}) gives an open book which is related to that shown in Figure 3 for $(Y_n, \eta^n_{i,j})$ by Hopf stabilization. We showed in Lemma 4.5 that the braid relation does not change how horizontal knots on the page are embedded up to an ambient contact isotopy, and thus after surgery and applying braid relation the image of the Legendrian knot $\mathcal{L}_{l,r}$ from ξ_{i+1} sits on a page of the open book as $\mathcal{L}_{l,r}$ in ξ_i. $\square$

4.2. **The computation of the invariants.**

Lemma 4.7. *All tight contact structures ξ_i on Y_∞ are homotopic and have 3-dimensional homotopy invariant $\theta(\xi_i) = -4$*

Proof. The homotopy between all the contact structures ξ_i was proved in [8, Proposition 2] therefore we can compute the 3-dimensional homotopy invariant on ξ_0, which has a Stein filling (V_∞, J_∞) obtained by attaching a Stein handle on a Legendrian right-handed trefoil knot with Thurston–Bennequin invariant 0 (see Figure 1). It is easy to see that $c_1(J_\infty) = 0$, $\chi(V_\infty) = 2$, and $\sigma(V_\infty) = 0$, therefore the formula for the 3-dimensional homotopy invariant in [9, Definition 4.2] yields $\theta(\xi_0) = -4$. $\square$

Lemma 4.8 ([5, Theorem 3.12])**.** *All tight contact structures $\eta_{i,j}$ are homotopic and have 3-dimensional homotopy invariant $\theta(\xi_i) = -6$.*

The reference computes $\frac{\theta}{4}$ for $i = 0$ or $i = n$, but the proof can be extended to all cases without modification.

Lemma 4.9. *$\underline{\widehat{HF}}(-Y_\infty) \cong \mathbb{Z}[H^1(-Y_\infty)] \oplus \mathbb{Z}[H^1(-Y_\infty)]$ with one summand in degree $\frac{1}{2}$ and one in degree $\frac{3}{2}$. Moreover $\underline{c}(\xi_0)$ is a generator of the summand in degree $\frac{1}{2}$.*

Proof. $-Y_\infty$ can be obtained by 0-surgery on the left-handed trefoil knot, and the Poincaré sphere $\Sigma(2,3,5)$ can be obtained by (-1)-surgery on the same knot, therefore the surgery exact triangle of Theorem 3.3 gives:

$$\begin{tikzcd} \widehat{HF}(S^3)[t,t^{-1}] \arrow[rr, "F"] & & \widehat{HF}(\Sigma(2,3,5))[t,t^{-1}] \arrow[dl] \\ & \underline{\widehat{HF}}(-Y_\infty) \arrow[ul] & \end{tikzcd}$$

The horizontal map F is induced by a cobordism W constructed by the attachment of a 2-handle with framing -1 along the left-handed trefoil knot, therefore the integer homology group $H_2(W)$ is generated by a surface $\widehat{\Sigma}$ with self-intersection $\widehat{\Sigma}^2 = -1$. The Spin^c-structures on W are indexed by integers k such that $\langle c_1(\mathfrak{s}_k), [\widehat{\Sigma}]\rangle = 2k+1$, so $c_1(\mathfrak{s}_k)^2 = -(2k+1)^2$. By Theorem 3.3, $F = \sum\limits_{k\in\mathbb{Z}} F_{W,\mathfrak{s}_k} \otimes t^k$. For any Spin^c-structure $\mathfrak{s}_k$ the map $F_{W,\mathfrak{s}_k}$ shifts the degree by $\frac{1}{4}\big(c_1(\mathfrak{s}_k)^2 - 2\chi(W) - 3\sigma(W)\big) = -k(k+1)$. Since $-k(k+1) \leq 0$, $\widehat{HF}(S^3) \cong \mathbb{Z}_{(0)}$ and $\widehat{HF}(\Sigma(2,3,5)) \cong \mathbb{Z}_{(2)}$, the horizontal map is trivial. This implies that $\underline{\widehat{HF}}(-Y_\infty) \cong \mathbb{Z}[H^1(-Y_\infty)] \oplus \mathbb{Z}[H^1(-Y_\infty)]$. The second homology group of Y_∞ is generated by an embedded torus, therefore the adjunction inequality [21, Theorem 7.1], which holds also for Heegaard Floer homology with twisted coefficients, implies that $\underline{\widehat{HF}}(-Y_\infty)$ is concentrated in the trivial Spin^c-structure. The Heegaard Floer homology groups for Spin^c-structures with torsion first Chern class admit an absolute $\mathbb{Q}$-grading [18, Section 7], which we are now going to determine for $\underline{\widehat{HF}}(-Y_\infty)$.

The map $\widehat{HF}(\Sigma(2,3,5))[t,t^{-1}] \to \underline{\widehat{HF}}(-Y_\infty)$ is induced by a 2-handle attachment with framing 0 because the first Betti number increases, therefore it has degree $-\frac{1}{2}$. The map $\underline{\widehat{HF}}(-Y_\infty) \to \widehat{HF}(S^3)[t,t^{-1}]$ is induced by a 2-handle attachment along a homologically non-trivial knot, therefore it also has degree $-\frac{1}{2}$; see for example [19, Lemma 3.1]. This implies that $\underline{\widehat{HF}}(-Y_\infty) \cong \mathbb{Z}[H^1(-Y_\infty)] \oplus\ Z[H^1(-Y_\infty)]$ with one summand in degree $\frac{3}{2}$ and the other one in degree $\frac{1}{2}$.

The contact invariant $c(\xi_0)$ has degree $-\frac{\theta(\xi_0)}{4} - \frac{1}{2} = \frac{1}{2}$ and is a generator of $\underline{\widehat{HF}}_{1/2}(-Y_\infty)$ by [20, Theorem 4.2]. ☐

Lemma 4.10 ([23])**.** *For any $n \geq 2$, $\widehat{HF}_{+1}(-Y_n) \cong \mathbb{Z}^{n-1}$ and the contact invariants $c(\eta_{0,-n+2})$, …, $c(\eta_{0,n-2})$ form a basis.*

The surgery described in Proposition 4.1 produces a cobordism Z_n from Y_∞ to Y_n which can be decomposed in two different ways:

- either as a cobordism W_∞ from Y_∞ to itself followed by a cobordism V_n from Y_∞ to Y_n if we attach 2-handles along $\mathcal{C}$ first, and then along $\mathcal{L}$,
- or as a cobordisms V_{n+1} from Y_∞ to Y_{n+1} followed by a cobordism W_n from Y_{n+1} to Y_n if we attach 2-handles along $\mathcal{L}$ first, and then along $\mathcal{C}$.

These cobordisms induce maps on Heegaard Floer homology according to Theorem 3.1. Now we compute the change of the coefficient group for the maps induced by the cobordisms above. Let $K(V_n) = \ker\big(H^2(V_n, Y_\infty) \to H^2(V_n)\big)$. By the cohomology exact sequence the map $H^1(Y_\infty) \to H^2(V_n, Y_\infty)$ is an isomorphism, therefore $K(V_n) \cong H^1(Y_\infty)$ and we can identify $\mathbb{Z}(V_n) = \mathbb{Z}[K(V_n)]$ with $\mathbb{Z}[H^1(Y_\infty)]$. Here we have used the fact that Y_n is an integer homology sphere. The manifolds Y_n are integer homology spheres, therefore $K(W_n)$ is the trivial group. Moreover $H^2(Z_n, \partial Z_n) = H^2(W_n, \partial W_n) \oplus H^2(V_{n+1}, \partial V_{n+1})$ and $H^2(Z_n) = H^2(W_n) \oplus H^2(V_{n+1})$, therefore $K(Z_n) \cong K(W_n) \oplus K(V_{n+1}) \cong K(V_{n+1})$.

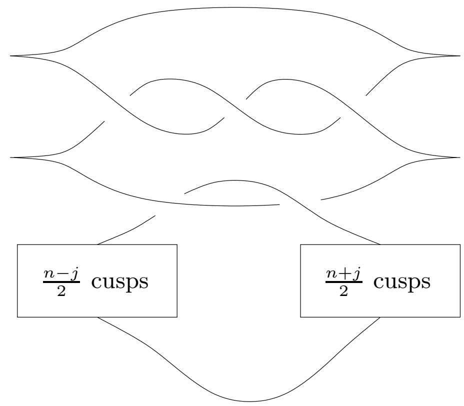

FIGURE 9. Legendrian surgery presentation of $(Y_n, \eta^n_{0,j})$.

Lemma 4.11. *The connecting homomorphism $\delta\colon H^1(Y_\infty) \to H^2(Z_n)$ in the Mayer–Vietoris sequence for the decomposition $Z_n = W_\infty \cup_{Y_\infty} V_n$ is the trivial map.*

Proof. It is easier to see this by taking the Poincaré–Lefschetz duals and looking at the associated map in the Mayer-Vietoris for relative homology. There $\delta\colon H^1(Y_\infty) \to H^2(Z_n)$ becomes $i\colon H_2(Y_\infty) \to H_2(Z_n, \partial Z_n)$. This map is trivial because the torus generating $H_2(Y_\infty)$ is homologous (indeed isotopic) to the torus generating the second homology group of the copy of Y_∞ in the boundary of Z_n. This can be seen by examining W_∞, which is built from Y_∞ by adding 2-handles along curves, each lying on a torus fiber. Hence the torus fibers in each boundary component of W_∞ are isotopic. □

Lemma 4.11 and Lemma 3.4 imply that $K(V_n)(W_\infty) \cong K(Z_n)$, so also $\mathbb{Z}[K(W_\infty)] \cong \mathbb{Z}[H^1(Y_\infty)]$. Moreover the cobordism W_∞ satisfies the hypotheses of Remark 3.2. By Theorem 3.1 and Remark 3.2 V_n and W_∞ induce maps

$$F_{V_n,\mathfrak{k}}\colon \widehat{HF}(-Y_n) \to \underline{\widehat{HF}}(-Y_\infty) \quad \text{and} \quad F_{W_\infty,\mathfrak{k}}\colon \underline{\widehat{HF}}(-Y_\infty) \to \underline{\widehat{HF}}(-Y_\infty).$$

By an abuse of notation, we will denote the canonical Spin^c-structure on a symplectic cobordism by $\mathfrak{k}$ regardless of what the cobordism or the symplectic form are; in fact this will not be very important in our proof. These maps fit into a diagram:

(3)
$$\begin{tikzcd}
\widehat{HF}(-Y_n) \arrow[r, "F_{W_n,\mathfrak{k}}"] \arrow[d, "F_{V_n,\mathfrak{k}}"] & \widehat{HF}(-Y_{n+1}) \arrow[d, "F_{V_{n+1},\mathfrak{k}}"] \\
\underline{\widehat{HF}}(-Y_\infty) \arrow[r, "F_{W_\infty,\mathfrak{k}}"] & \underline{\widehat{HF}}(-Y_\infty)
\end{tikzcd}$$

which commutes for a suitable choice of the maps in their equivalence class, because Lemma 4.11 implies that the restriction map $\mathfrak{s} \mapsto (\mathfrak{s}|_{W_\infty}, \mathfrak{s}|_{V_n})$ gives an isomorphism $\mathrm{Spin}^c(Z_n) \cong \mathrm{Spin}^c(W_\infty) \times \mathrm{Spin}^c(V_n)$, and we have a similar isomorphism $\mathrm{Spin}^c(Z_n) \cong \mathrm{Spin}^c(V_{n+1}) \times \mathrm{Spin}^c(W_\infty)$ because $H^1(Y_n) = 0$.

From now on we will drop the canonical Spin^c-structure $\mathfrak{k}$ from the notation and make a change in the coefficient ring which will allow us to write our formulas in a more symmetric form. Let $\Lambda = \mathbb{Z}[H^1(Y_\infty, \frac{1}{2}\mathbb{Z})]$ with the $\mathbb{Z}[H^1(Y_\infty)]$-module structure defined by the inclusion $H^1(Y_\infty) \subset H^1(Y_\infty, \frac{1}{2}\mathbb{Z})$. Since Λ is a free module over $\mathbb{Z}[H^1(Y_\infty)]$, we have

$$\underline{\widehat{HF}}(-Y_\infty, \Lambda) \cong \underline{\widehat{HF}}(-Y_\infty) \otimes_{H^1(-Y_\infty)} \Lambda \cong \Lambda_{(\frac{1}{2})} \oplus \Lambda_{(\frac{3}{2})}.$$

We choose an identification $\widehat{\underline{HF}}_{\frac{1}{2}}(-Y_\infty, \Lambda) \cong \Lambda$ such that $[c(\xi_0)]$ corresponds to $[1]$.

Lemma 4.12. *We can choose a representative of F_{V_n}, an identification of $\widehat{\underline{HF}}_{(\frac{1}{2})}(-Y_\infty, \Lambda)$ with $\mathbb{Z}[t^{\pm\frac{1}{2}}]$, and signs for the contact invariants $c(\eta_{0,j})$ such that*

$$F_{V_n}(c(\eta^n_{0,j})) = t^{j/2}.$$

Proof. Let us view the 4-manifold V_∞ used in the proof of Lemma 4.7, constructed by adding a 2-handle to D^4 along the right-handed trefoil knot in Figure 1 with attaching framing 0, as a cobordism from $-Y_\infty$ to S^3, and let $X_n = V_\infty \cup_{Y_\infty} V_n$. The second homology group of X_n is generated by the class T of a torus fibre in Y_∞ and by the class of a sphere S such that $S \cdot T = 1$. Let $\mathfrak{s}_j$ be the Spin^c-structure on X_n such that $\langle c_1(\mathfrak{s}_j), T\rangle = 0$ and $\langle c_1(\mathfrak{s}_j), S\rangle = j$ with $j \equiv n \pmod 2$. There is a generator h of $H^1(Y_\infty)$ such that $\mathfrak{s}_{j+1} = \mathfrak{s}_j + \delta(h)$, and we identify Λ with $\mathbb{Z}[t^{\pm 1/2}]$ by sending e^h to t. By the composition formula 3.5 we can choose F_{V_n} such that

$$F_{X_n,\mathfrak{s}_j} = \Pi \circ F_{V_\infty} \circ t^{-j/2} \cdot F_{V_n}.$$

The restriction map $H^2(V_\infty; \mathbb{Z}) \to H^2(Y_\infty; \mathbb{Z})$ is an isomorphism and $H^1(V_\infty) = 0$, therefore $K(V_\infty) \cong H^1(Y_\infty)$, so V_∞ induces a map

$$F_{V_\infty} : \widehat{\underline{HF}}(-Y_\infty, \Lambda) \to \widehat{HF}(S^3)[t^{1/2}, t^{-1/2}].$$

Since the right-handed trefoil knot has a Legendrian representative with Thurston–Bennequin invariant $+1$, V_∞ can be endowed with a Stein structure providing a Stein cobordism from (S^3, ξ_{st}) to (Y_∞, ξ_0), so $[F_{V_\infty}(\underline{c}(\xi_0))] = [c(\xi_{st})]$. Then, after identifying both $\widehat{\underline{HF}}_{\frac{1}{2}}(-Y_\infty, \Lambda)$ and $\widehat{HF}(S^3)[t^{1/2}, t^{-1/2}]$ with $\mathbb{Z}[t^{\pm 1/2}]$, we can choose F_{V_∞} to be the conjugation map $t \mapsto t^{-1}$.

The Spin^c-structure $\mathfrak{s}_j$ on X_n is the canonical Spin^c-structure of the Stein filling (X_n, J_n) of $\eta^n_{0,j}$ described by the Legendrian surgery diagram in Figure 9, then we know from [23] that $F_{X_n,\mathfrak{s}_j}(c(\eta^m_{0,j})) = c(\xi_{st})$, and $F_{X_n,\mathfrak{s}_k}(c(\eta^n_{0,j})) = 0$ for $k \neq j$. Using the composition formula and the fact that F_{V_∞} is, in our choice of identifications, the conjugation map, we conclude that $F_{V_n}(c(\eta^n_{0,j})) = t^{j/2}$. □

Now we choose the maps in Diagram (3) so that it becomes commutative. The horizontal map in the upper part is fixed because the Y_n are integer homology spheres, while the vertical maps are fixed by the choices in Lemma 4.12.

Lemma 4.13. *If we choose F_{V_n} such that $F_{V_n}(c(\eta^n_{0,j})) = t^{j/2}$ for all n, then Diagram (3) commutes if we choose the map F_{W_∞} to be represented by the multiplication by $t^{\frac{1}{2}} - t^{-\frac{1}{2}}$.*

Proof. The contact structure ξ_1 is obtained from ξ_0 by a generalized Lutz twist, therefore $[c(\xi_1)] = [(t-1)c(x_0)]$ by [6, Theorem 2]. This implies that F_{W_∞} is the multiplication by $(t-1)t^{k/2}$ for some $k \in \mathbb{Z}$. We will now determine which choice for F_{W_∞} will make Diagram (3) commutative.

Inverting the orientation of the contact planes results in a symmetry of the triangle (1) about its vertical axis. In particular the contact structure $\eta^n_{n-2,0}$ is invariant under conjugation, and $\eta^n_{0,j}$ is conjugated to $\eta^n_{0,-j}$; see [4, Proposition 3.8], where $\eta^n_{n-2,0}$ is called η_0, and $\eta^n_{0,j}$ is called η_j. In the reference only odd n are considered, but the proof carries through in general. We have already seen that $c(\eta^n_{n-2,0})$ can be expressed as a linear combination of the elements $c(\eta^n_{n-2,0})$. The invariance of $c(\eta^n_{n-2,0})$ by conjugation implies that $c(\eta^n_{n-2,0}) = a_{1-n}c(\eta^n_{0,1-n}) + \ldots + a_{n-1}c(\eta^n_{0,n-1})$ with $a_j = a_{-j}$, hence $F_{V_n}(c(\eta^n_{n-2,0}))$ is a symmetric Laurent polynomial in the variable $t^{1/2}$ because it is invariant under the automorphism $t^{1/2} \mapsto t^{-1/2}$.

Since $F_{V_{n+1}}(c(\eta^n_{n-2,0})) = c(\eta^{n+1}_{n-1,0})$, the composite $F_{V_{n+1}} \circ F_{W_n}$ maps $c(\eta^n_{n-2,0})$ to a symmetric Laurent polynomial too. If Diagram (3) commutes, then F_{W_∞} maps a symmetric Laurent polynomials to a symmetric Laurent polynomials, therefore it must be the multiplication by $t^{1/2} - t^{-1/2}$. □

Proof of Theorem 1.3. The theorem will be proved by induction on n. The initial step is $n = 2$. Since there is a unique tight contact structure on Y_2 by [7, Theorem 4.9], there is nothing to prove in this case. Now we assume that Formula (2) holds for the tight contact structures on Y_n, for some n, and we prove that this implies that Formula (2) holds for the tight contact structures on Y_{n+1}. From the surgery construction we have

$$F_{W_n}(c(\eta^n_{i,j})) = c(\eta^{n+1}_{i+1,j}),$$

and the induction hypothesis gives, on Y_{n+1}, the following expression for the contact invariants of $\eta^{n+1}_{i,j}$ for $i \geq 1$ in terms of the contact invariants of $\eta^{n+1}_{1,j}$:

$$c(\eta^{n+1}_{i+1,j}) = \sum_{k=0}^{i} (-1)^k \binom{i}{k} c(\eta^{n+1}_{1,j-i+2k}). \tag{4}$$

We can compute $c(\eta^{n+1}_{1,j})$ by the the commutativity of Diagram 3: in fact

$$F_{V_{n+1}}(c(\eta^{n+1}_{1,j})) = F_{W_\infty}(F_{V_n}(c(\eta^n_{0,j}))) = t^{(j+1)/2} - t^{(j-1)/2},$$

therefore

$$c(\eta^{n+1}_{1,j}) = c(\eta^{n+1}_{0,j+1}) - c(\eta^{n+1}_{0,j-1}) \tag{5}$$

because the map $F_{V_{n+1}}$ is injective.

If we substitute $c(\eta^{n+1}_{1,j})$ in Equation 4 with the right-hand side of Equation 5, and write $j - i + 1 + 2k = j - (i + 1) + 2(k + 1)$ we obtain:

$$\begin{aligned} c(\eta^{n+1}_{i+1,j}) &= \sum_{k=0}^{i} (-1)^k \binom{i}{k} (c(\eta^{n+1}_{0,j-(i+1)+2(k+1)}) - c(\eta^{n+1}_{0,j-(i+1)+2k})) \\ &= \sum_{k=1}^{i+1} (-1)^{k-1} \binom{i}{k-1} c(\eta^{n+1}_{0,j-(i+1)+2k}) - \sum_{k=0}^{i} (-1)^k \binom{i}{k} c(\eta^{n+1}_{0,j-(i+1)+2k}) \\ &= \sum_{k=0}^{i+1} (-1)^{k-1} \left[\binom{i}{k} + \binom{i}{k-1} \right] c(\eta^{n+1}_{0,j-(i+1)+2k}) \\ &= \sum_{k=0}^{i+1} (-1)^{k-1} \binom{i+1}{k} c(\eta_{0,j-(i+1)+2k}) \end{aligned}$$

from which the statement of Theorem 1.3 follows. □

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CNRS, Université de Nantes, FRANCE
E-mail address: paolo.ghiggini@univ-nantes.fr

American Institute of Mathematics, Palo Alto, CA
E-mail address: jvanhorn@aimath.org